Sunflowers and symmetric designs*

Brahadeesh Sankarnarayanan

Indian Institute of Technology Bombay
bs@math.iitb.ac.in

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*Based on joint works with N. Balachandran, S. Das, R. Mathew, K.V. Kher

Fractional θ-intersecting families

Definition (Balachandran–Mathew–Mishra 2019)

Let $0 < \theta < 1$ be a rational. A collection $\mathcal{F}$ of subsets of $[n]$ is a (fractional) θ-intersecting family if for all $A, B \in \mathcal{F}$, $A \neq B$, we have

$$|A \cap B| \in \{\theta|A|, \theta|B|\}.$$

Example 1

Let $\theta = 1/2$.
For $n = 8$, consider the family

$$\{12, 13, 14, 15, 16, 17, 18\}$$

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This is a $\frac{1}{2}$-intersecting family over [8] containing 10 sets.

Example 2

Consider the matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

This is a 4×4 Hadamard matrix: it has entries in $\{\pm 1\}$ and the rows are pairwise orthogonal.

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This is a 4×4 Hadamard matrix: it has entries in $\{\pm 1\}$ and the rows are pairwise orthogonal.

View each row as the $\{\pm 1\}$-incidence vector of a subset of $[4]$.

This defines a $\frac{1}{2}$-intersecting family.

Example 2

Next, consider the block matrix

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix},$$

where J is the all-ones matrix.

Example 2

$$
\left[\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & -1
\end{array}\right]
$$

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1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & -1
\end{array}\right]
\qquad
\begin{array}{l}
12345678 \\
1357 \\
1256 \\
1458 \\
1234 \\
1368 \\
1278 \\
1467 \\
1234 \\
13 \\
12 \\
14
\end{array}
$$

Example 2

$$\left[\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
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1 & -1 & -1 & 1 & -1 & -1 & -1 & -1
\end{array}\right]$$

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1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\hline
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12 \\
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\end{array}
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1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
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1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
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1 & -1 & -1 & 1 & -1 & -1 & -1 & -1
\end{array}\right]
$$

~~12345678~~
1357
1256
1458
1234
1368
1278
1467
~~1234~~
13
12
14

Example 2

1	1	1	1	1	1	1	1	~~12345678~~
1	−1	1	−1	1	−1	1	−1	1357
1	1	−1	−1	1	1	−1	−1	1256
1	−1	−1	1	1	−1	−1	1	1458
1	1	1	1	−1	−1	−1	−1	1234
1	−1	1	−1	−1	1	−1	1	1368
1	1	−1	−1	−1	−1	1	1	1278
1	−1	−1	1	−1	1	1	−1	1467
1	1	1	1	−1	−1	−1	−1	~~1234~~
1	−1	1	−1	−1	−1	−1	−1	13
1	1	−1	−1	−1	−1	−1	−1	12
1	−1	−1	1	−1	−1	−1	−1	14

This is also a $\frac{1}{2}$-intersecting family over [8] containing 10 sets.

Example (Sunflower family)

Let $\mathcal{F}_s := \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}$.
Then, $\mathcal{F}_s$ is $\frac{1}{2}$-intersecting, and $|\mathcal{F}_s| = \frac{3n}{2} - 2$.

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Example (Hadamard family)

Let H be an $m \times m$ Hadamard matrix in normal form, and let J be the $m \times m$ all-ones matrix. Let $A_1, \ldots, A_{3m}$ be the rows of

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix},$$

viewed as the $\{\pm 1\}$-incidence vectors of subsets of $[2m]$.

Then, $\mathcal{F}_H := \{A_i : i \in [3m] \setminus \{1, 2m+1\}\}$ is a $\frac{1}{2}$-intersecting family. Writing $2m = n$, we have $|\mathcal{F}_H| = \frac{3n}{2} - 2$.

Are these families extremal?

Even a linear upper bound is not known!

Theorem (Balachandran–Mathew–Mishra 2019)

Let $\theta \in (0,1) \cap \mathbb{Q}$. If $\mathcal{F}$ is a θ-intersecting family over $[n]$, then

$$|\mathcal{F}| \leq O_\theta(n \log(n)^2 \log\log(n)).$$

Conjecture (Balachandran–Mathew–Mishra 2019)

Let $\theta \in (0,1) \cap \mathbb{Q}$. There is a constant $c > 0$ such that any θ-intersecting family over $[n]$ has size at most cn.

A closer look at the two examples (Round 1)

These two examples are at the extreme ends of a tower of *hierarchically* $\frac{1}{2}$-intersecting families.

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- In the sunflower family

$$\mathcal{F}_s = \{12, 13, 14, 15, 16, 17, 18, 1234, 1256, 1278\}$$

for any $r \geq 2$ and any pairwise distinct $A_1, \ldots, A_r \in \mathcal{F}_s$ we have $|A_1 \cap \cdots \cap A_r| \in \left\{\frac{1}{2}|A_1|, \ldots, \frac{1}{2}|A_r|\right\}$.

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- In the Hadamard family

$$\mathcal{F}_H = \{1357, 1256, 1458, 1234, 1368, 1278, 1467, 13, 12, 14\}$$

  this property is not satisfied even for $r = 3$.

Hierarchically r-closed fractional θ-intersecting families

Definition

Let $r \geq 2$ and $\theta \in (0,1) \cap \mathbb{Q}$. A family $\mathcal{F}$ of subsets of $[n]$ is called hierarchically r-closed θ-intersecting if, for each $2 \leq t \leq r$ and any t distinct sets $A_1, \ldots, A_t$ in $\mathcal{F}$ we have

$$\left| \bigcap_{i=1}^{t} A_i \right| \in \{\theta |A_i| : 1 \leq i \leq t\}.$$

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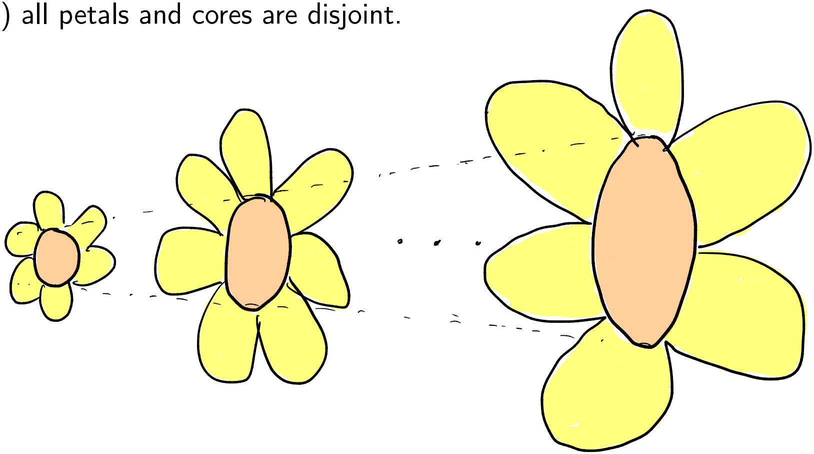
$$\left| \bigcap_{i=1}^{t} A_i \right| \in \{\theta |A_i| : 1 \leq i \leq t\}.$$

Question

What is the maximum size of a hierarchically r-closed θ-intersecting family over $[n]$, when $r \geq 3$?

Bouquets of sunflowers

A bouquet of sunflowers is a family of subsets of $[n]$ such that:
(1) each level is a sunflower; (2) cores form an increasing chain;
(3) all petals and cores are disjoint.



$\mathcal{F}_s$ is (nearly) a bouquet

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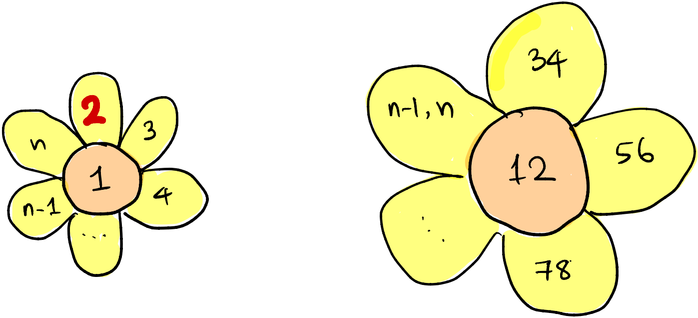
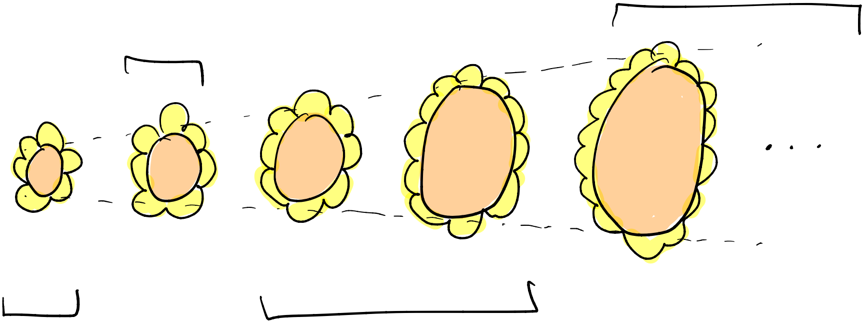


Hierarchically θ-intersecting families contain large bouquets

Key idea: if $|A| < \theta|B|$, then $|A \cap B| = \theta|A|$.
So, "dyadically" bunch up the sunflowers to get an upper bound on $|\mathcal{F}|$.



Hierarchically θ-intersecting families are linear in size!

Theorem
(Balachandran–Bhattacharya–Kher–Mathew–S. 2023)

There is a constant $c_\theta \leq \frac{3}{2}$ such that, if $\mathcal{F}$ is an r-closed θ-intersecting family over $[n]$ with $r \geq 3$, then $|\mathcal{F}| \leq c_\theta n$.

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When $\theta = 1/2$,

$$|\mathcal{F}| \leq \lfloor \frac{3n}{2} \rfloor - 2$$

for all $n \geq 2$.

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Moreover:

- *Any family* $\mathcal{F}$ *that attains this bound is just* $\sigma(\mathcal{F}_s)$ *for some permutation* σ *of* $[n]$*.*

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Moreover:

- *Any family $\mathcal{F}$ that attains this bound is just $\sigma(\mathcal{F}_s)$ for some permutation σ of $[n]$.*
- *There exists an absolute constant $C > 0$ such that the following holds: if $|\mathcal{F}| \geq (\frac{3}{2} - \epsilon)n$ for some $0 < \epsilon < 0.1$, then for some permutation σ of $[n]$, $|\sigma(\mathcal{F}) \setminus \mathcal{F}_s| < C\epsilon n$.*

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The story so far ... and a follow-up question

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Question

If the sets in a θ-intersecting family $\mathcal{F}$ are not "too large", then is the size of $\mathcal{F}$ linear in n?

A theorem of Deza

Say that a family $\mathcal{F}$ is w-bounded if all the sets in $\mathcal{F}$ have size at most w.

Theorem (Deza 1974)

Let $\mathcal{F}$ be a w-bounded family of subsets of $[n]$ such that all pairwise intersections have the same cardinality.
If $|\mathcal{F}| \geq w^2 - w + 2$, then $\mathcal{F}$ is a sunflower.

Bounded θ-intersecting families . . .

Proposition (Balachandran–Das–S. 2024)

Let $\mathcal{F}$ be a w-bounded θ-intersecting family over $[n]$. Then, there is a bouquet $\mathcal{B}$ in $\mathcal{F}$ such that $|\mathcal{F} \setminus \mathcal{B}| \leq w^3$.

Bounded θ-intersecting families ...

Proposition (Balachandran–Das–S. 2024)

Let $\mathcal{F}$ be a w-bounded θ-intersecting family over $[n]$. Then, there is a bouquet $\mathcal{B}$ in $\mathcal{F}$ such that $|\mathcal{F} \setminus \mathcal{B}| \leq w^3$.

Thus, if w is not "too large", then a w-bounded θ-intersecting family $\mathcal{F}$ contains a large bouquet.

We can also modify the arguments from the hierarchical setting to get a bound on the size of such a bouquet by a double-counting argument.

Bounded θ-intersecting families are linear in size!

Theorem (Balachandran–Das–S. 2024)

If $w \leq O(n^{1/3})$ then there is a constant $C > 0$ such that the following holds: for all sufficiently large n, if $\mathcal{F}$ is a w-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| \leq Cn$.

Theorem (Balachandran–Das–S. 2024)

If $\mathcal{F}$ is a $o(n^{1/3})$-bounded $\frac{a}{b}$-intersecting family over $[n]$, then $|\mathcal{F}| \leq (C_\theta + o(1))n$, where $C_\theta = \frac{1}{b-a}\sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$.

The constant is tight for $\theta \in \{1/3\} \cup [1/2, 1)$.

A closer look at the two examples (Round 2)

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Both $\mathcal{F}_s$ and $\mathcal{F}_H$ have sets of only two distinct sizes:
2 and 4 for $\mathcal{F}_s$, and $n/2$ and $n/4$ for $\mathcal{F}_H$.

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A trivial upper bound on the size of a θ-intersecting family over $[n]$ having sets of only two distinct sizes is $2n$.

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2 and 4 for $\mathcal{F}_s$, and $n/2$ and $n/4$ for $\mathcal{F}_H$.

A trivial upper bound on the size of a θ-intersecting family over $[n]$ having sets of only two distinct sizes is $2n$.

But $\mathcal{F}_s$ and $\mathcal{F}_H$ have size $\frac{3n}{2} - 2$.

A closer look at the two examples (Round 2)

Question

Can the trivial upper bound of $2n$ be improved when $\mathcal{F}$ has sets of only two distinct sizes?

A matrix associated to $\mathcal{F}_s$ over [8]

$$\mathcal{F}_s = \{12, 13, 14, 15, 16, 17, 18, 1234, 1256, 1278\}$$

$$X = \left[\begin{array}{rrrrrrrr}
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]$$

A matrix associated to $\mathcal{F}_s$ over [8]

$$\mathcal{F}_s = \{12, 13, 14, 15, 16, 17, 18, 1234, 1256, 1278\}$$

$$XX^T = \left[\begin{array}{ccccccc|ccc}
8 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 8 & 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 \\
4 & 4 & 8 & 4 & 4 & 4 & 4 & 4 & 0 & 0 \\
4 & 4 & 4 & 8 & 4 & 4 & 4 & 0 & 4 & 0 \\
4 & 4 & 4 & 4 & 8 & 4 & 4 & 0 & 4 & 0 \\
4 & 4 & 4 & 4 & 4 & 8 & 4 & 0 & 0 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 8 & 0 & 0 & 4 \\
\hline
4 & 4 & 4 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
4 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 8 & 0 \\
4 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 8
\end{array}\right]$$

A matrix associated to $\mathcal{F}_s$ over [8]

$$\mathcal{F}_s = \{12, 13, 14, 15, 16, 17, 18, 1234, 1256, 1278\}$$

$$M = (8J - XX^T)/2$$

$$= \left[\begin{array}{ccccccc|ccc} 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 & 4 \\ 2 & 2 & 2 & 2 & 0 & 2 & 2 & 4 & 2 & 4 \\ 2 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 4 & 4 & 2 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 \\ 2 & 4 & 4 & 2 & 2 & 4 & 4 & 4 & 0 & 4 \\ 2 & 4 & 4 & 4 & 4 & 2 & 2 & 4 & 4 & 0 \end{array}\right]$$

A matrix associated to $\mathcal{F}_H$ over [8]

$$\mathcal{F}_H = \{1357, 1256, 1458, 1234, 1368, 1278, 1467, 13, 12, 14\}$$

$$M = (8J - XX^T)/2$$

$$= \left[\begin{array}{ccccccc|ccc} 0 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 4 & 4 \\ 4 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 4 \\ 4 & 4 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 2 \\ 4 & 4 & 4 & 0 & 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 0 & 4 & 4 & 2 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 2 \\ \hline 2 & 4 & 4 & 2 & 2 & 4 & 4 & 0 & 2 & 2 \\ 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 0 & 2 \\ 4 & 4 & 2 & 2 & 4 & 4 & 2 & 2 & 2 & 0 \end{array}\right]$$

Low-rank symmetric matrices with zero diagonal

- By these constructions, we get $n \times n$ matrices of rank $\approx 2n/3$ (since the families $\mathcal{F}_s$ and $\mathcal{F}_H$ have size $\approx 3n/2$).
- Similarly, if there are $\frac{1}{2}$-intersecting families over $[n]$ of size $2n$, then we will get $n \times n$ matrices of rank $n/2$.

Low-rank symmetric matrices with zero diagonal

- By these constructions, we get $n \times n$ matrices of rank $\approx 2n/3$ (since the families $\mathcal{F}_s$ and $\mathcal{F}_H$ have size $\approx 3n/2$).
- Similarly, if there are $\frac{1}{2}$-intersecting families over $[n]$ of size $2n$, then we will get $n \times n$ matrices of rank $n/2$.

How low can the rank of such matrices be?

- Symmetric
- Zero diagonal
- Off-diagonal entries are nonzero and either α or β
- Has an $(m+n) \times (m+n)$ block form.

Denote the collection of all such matrices by $\mathrm{Sym}(\alpha^{(m)}, \beta^{(n)})$.

Sym($\alpha^{(m)}, \beta^{(n)}$) and bipartite graphs

$\mathcal{F}_s = \{12, 13, 14, 15, 16, 17, 1234, 1256, 1278\}$

$$\left[\begin{array}{ccccccc|ccc}
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 \\
2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 & 4 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 & 4 \\
2 & 2 & 2 & 2 & 0 & 2 & 2 & 4 & 2 & 4 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 4 & 4 & 2 \\
\hline
2 & 2 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 \\
2 & 4 & 4 & 2 & 2 & 4 & 4 & 4 & 0 & 4 \\
2 & 4 & 4 & 4 & 4 & 2 & 2 & 4 & 4 & 0
\end{array}\right]$$

Sym($\alpha^{(m)}, \beta^{(n)}$) and bipartite graphs

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$$\left[\begin{array}{ccccccc|ccc}
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
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2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 & 4 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 & 4 \\
2 & 2 & 2 & 2 & 0 & 2 & 2 & 4 & 2 & 4 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 4 & 4 & 2 \\
\hline
2 & 2 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 \\
2 & 4 & 4 & 2 & 2 & 4 & 4 & 4 & 0 & 4 \\
2 & 4 & 4 & 4 & 4 & 2 & 2 & 4 & 4 & 0
\end{array}\right]$$

Let the α denote incidence and β denote non-incidence in the off-diagonal blocks of M.
Then, we get an associated bipartite graph G_M.

Multiplicity of eigenvalues vs. ranks of matrices

Theorem (Balachandran–S. 2024)

Let $M \in \mathrm{Sym}(\alpha^{(m)}, \beta^{(n)})$*. Let* $\mu(\alpha, \beta) \in \mathbb{C}$ *be given by*

$$\mu^2 = \frac{\alpha\beta}{(\alpha - \beta)^2}.$$

If ν *is the multiplicity of* μ *as an eigenvalue of* G_M*, then*

$$|\mathrm{rank}(M) - (m + n - \nu)| \leq 2.$$

A theorem of Rowlinson

Theorem (Rowlinson 2016)

Let G be a connected bipartite graph of order $n > 5$, with $\mu \notin \{-1, 0\}$ as an eigenvalue of multiplicity $\nu > 1$.

(a) *If d is the maximum degree in G, then $\nu \leq n - 1 - d$.*

(b) *If equality holds in (a), then $\nu \leq d - 1$.*

(c) *If equality holds in (b), then G is the bipolar cone over a graph G_0, where G_0 is either the incidence graph of a symmetric 2-design, or a 2-balanced bipartite graph.*

Symmetric designs

Definition

A symmetric 2-(v, k, λ) design Δ is a collection of k-subsets of $[v]$ such that every pair of elements in v belongs to exactly λ sets in the collection, and $|\Delta| = v$.

- Any symmetric 2-(v, k, λ) design Δ has an associated bipartite point-block incidence graph G_Δ, which has spectrum

$$\{v, (\sqrt{k-\lambda})^{(v-1)}, (-\sqrt{k-\lambda})^{(v-1)}, -v\}.$$

Low-rank matrices in $\mathrm{Sym}(\alpha^{(n)}, \beta^{(n)})$ over $\mathbb{R}$

Let $\beta := (3+\sqrt{5})/2$. For $n = 5$, we have

$$M = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 1 & 1 & 1 & \beta & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & \beta & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & \beta & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \beta & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \beta \\ \hline \beta & 1 & 1 & 1 & 1 & 0 & \beta & \beta & \beta & \beta \\ 1 & \beta & 1 & 1 & 1 & \beta & 0 & \beta & \beta & \beta \\ 1 & 1 & \beta & 1 & 1 & \beta & \beta & 0 & \beta & \beta \\ 1 & 1 & 1 & \beta & 1 & \beta & \beta & \beta & 0 & \beta \\ 1 & 1 & 1 & 1 & \beta & \beta & \beta & \beta & \beta & 0 \end{array}\right]$$

and $\mathrm{rank}(M) = 6$.

In general, we can find matrices $M_{2n} \in \mathrm{Sym}(1^{(n)}, \beta^{(n)})$ such that $\mathrm{rank}(M_{2n}) \leq n+3$. These matrices are constructed from the complete bipartite graph $K_{n,n}$ minus a perfect matching.

Low-rank matrices in $\mathrm{Sym}(\alpha^{(n)}, \beta^{(n)})$ over $\mathbb{Q}$ (or $\mathbb{Z}$)

Theorem (Balachandran–S. 2024)

For each $\varepsilon > 0$, there exists $c_\varepsilon \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ and $\beta_\varepsilon \in \mathbb{Z}$ such that there is a sequence of matrices $M_{2n} \in \mathrm{Sym}((\beta_\varepsilon - 1)^{(n)}, \beta_\varepsilon^{(n)})$ for which $\mathrm{rank}(M_{2n}) \leq c_\varepsilon n + O(1)$.

These matrices are constructed from Hadamard designs, which are symmetric 2-$(4n - 1, 2n - 1, n - 1)$ designs.

Ruling out candidates for low-rank matrices over $\mathbb{Z}$

Many of the known infinite families of symmetric 2-(v, k, λ) designs have the property that $k - \lambda$ is a prime power.

Almost none of these families are viable candidates for producing low rank matrices!

Proposition (Balachandran–S. 2024)

Let Δ be a symmetric 2-(v, k, λ) design with $k - \lambda = p^m$ for some prime p and integer $m \geq 1$. Consider $M_\Delta \in \mathrm{Sym}(\alpha^{(v)}, \beta^{(v)})$.

If $\mathrm{rank}(M_\Delta) \leq v + 3$, then $p^m = 2$.

The story so far . . .

We wanted to know whether the trivial upper bound of $2n$ for the size of a $\frac{1}{2}$-intersecting $\mathcal{F}$ can be improved when $\mathcal{F}$ has sets of only two sizes.

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We wanted to know whether the trivial upper bound of $2n$ for the size of a $\frac{1}{2}$-intersecting $\mathcal{F}$ can be improved when $\mathcal{F}$ has sets of only two sizes.

The previous results show that this is not possible for real and rational/integral matrices.

In fact, very few of the known infinite families of symmetric designs are helpful in finding low-rank matrices.

The story so far . . . and a followup question

But recall that the distinct sizes of the sets in $\mathcal{F}_s$ and $\mathcal{F}_H$ are both in the proportion $1 : 2$.

Question

Can we improve upon the trivial upper bound when $\mathcal{F}$ only has sets of two distinct sizes in the proportion $1 : 2$?

The Fano plane $PG(2,2)$

The Hadamard construction is longer helpful.

In fact, the 2-$(7,3,1)$ design (i.e., the Fano plane) is the *only* symmetric 2-design that produces matrices of low rank *and* which has entries in the proportion $1:2$ [Royle 2023].

This construction gives matrices $M_N \in \mathrm{Sym}(2^{(7n)}, 4^{(7n)})$ of rank at most $\frac{4N}{7}$ where $N = 14n$, improving the previously best-known bound of $\frac{2N}{3}$.

The matrix arising from the Fano plane for $N = 14$

$$
\left[\begin{array}{ccccccc|ccccccc}
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\
2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 2 & 2 & 4 & 4 \\
2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 \\
2 & 2 & 2 & 2 & 0 & 2 & 2 & 4 & 2 & 4 & 4 & 2 & 4 & 2 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 & 2 & 2 & 4 & 4 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 4 & 4 & 2 & 4 & 2 & 2 & 4 \\
\hline
2 & 2 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 4 & 4 & 4 & 4 \\
2 & 4 & 4 & 2 & 2 & 4 & 4 & 4 & 0 & 4 & 4 & 4 & 4 & 4 \\
2 & 4 & 4 & 4 & 4 & 2 & 2 & 4 & 4 & 0 & 4 & 4 & 4 & 4 \\
4 & 2 & 4 & 2 & 4 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 4 \\
4 & 2 & 4 & 4 & 2 & 4 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 \\
4 & 4 & 2 & 2 & 4 & 4 & 2 & 4 & 4 & 4 & 4 & 4 & 0 & 4 \\
4 & 4 & 2 & 4 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0
\end{array}\right]
$$

The matrix arising from the Fano plane for $N = 14$

$$
\left[\begin{array}{cccccccccc|cccc}
0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\
2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 2 & 2 & 4 & 4 \\
2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 \\
2 & 2 & 2 & 2 & 0 & 2 & 2 & 4 & 2 & 4 & 4 & 2 & 4 & 2 \\
2 & 2 & 2 & 2 & 2 & 0 & 2 & 4 & 4 & 2 & 2 & 4 & 4 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 4 & 4 & 2 & 4 & 2 & 2 & 4 \\
2 & 2 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 4 & 4 & 4 & 4 \\
2 & 4 & 4 & 2 & 2 & 4 & 4 & 4 & 0 & 4 & 4 & 4 & 4 & 4 \\
2 & 4 & 4 & 4 & 4 & 2 & 2 & 4 & 4 & 0 & 4 & 4 & 4 & 4 \\
\hline
4 & 2 & 4 & 2 & 4 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 4 \\
4 & 2 & 4 & 4 & 2 & 4 & 2 & 4 & 4 & 4 & 4 & 0 & 4 & 4 \\
4 & 4 & 2 & 2 & 4 & 4 & 2 & 4 & 4 & 4 & 4 & 4 & 0 & 4 \\
4 & 4 & 2 & 4 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0
\end{array}\right]
$$

The Fano family $\mathcal{F}_{\text{Fano}}$

For $N = 14$, we can actually construct a $\frac{1}{2}$-intersecting family $\mathcal{F}_{\text{Fano}}$ over $[8]$ of size 14:

$$\{12, 13, 14, 15, 16, 17, 18\}$$
$$\cup$$
$$\{1234, 1256, 1278\}$$
$$\cup$$
$$\{1357, 1368, 1458, 1467\}$$

Similar modifications can be used to get $\frac{1}{2}$-intersecting families over $[n]$ of size more than $\frac{3n}{2} - 2$ for $n \leq 15$.

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